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BRIEF COMMUNICATION

THE EINSTEIN VISCOSITY CORRECTION IN *n* DIMENSIONS

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Abstract—The effective viscosity of a dilute suspension of rigid n-dimensional hyperspheres in a viscous fluid at small particle Reynolds numbers is determined; the result being

$$\mu_{\text{eff}} = \mu\left(1+\frac{n+2}{2}\phi\right).$$

Expressions are also given for the n-dimensional Stokes velocity and pressure fields for a hypersphere in a pure straining flow.

INTRODUCTION

In teaching suspension mechanics as a topic in a course on transport processes, one is confronted with the difficulty of assigning a homework problem which is doable in a time scale small compared with that of a normal doctoral thesis. After having presented Einstein's correction to the viscosity of a dilute suspension of spheres, the natural homework problem is that for 2-dimensional rigid cylinders (a solution to Stokes equations existing in this case), and the n-dimensional case provides good practice with cartesian tensorial manipulations and the summation convention. Thus, the motivation for this brief communication.

We consider a force-free, couple-free hypersphere of radius a located at the origin in a pure straining motion with rate of strain tensor E_{ij}, which is symmetric and traceless. Making use of the linearity of the Stokes equations and the fact that the only vector present is $\mathbf{x}$ the position vector, and following Brenner (1981), we may write for the velocity and pressure fields in cartesian tensor notation with the summation convention

$$u_i(\mathbf{x}) = E_{ij}x_j + V_{ijk}E_{jk}\,, \tag{1}$$

$$p(\mathbf{x}) = p^{\infty} + P_{jk}E_{jk}\,. \tag{2}$$

Here $E_{ij}x_j$ and p^{∞} are the undisturbed velocity and pressure fields at infinity, and V_{ijk} and P_{jk}, being purely geometric, are given by

$$V_{ijk} = -\frac{1}{2}(\delta_{ij}x_k + \delta_{ik}x_j)\left(\frac{a}{r}\right)^{(n+2)} - \frac{(n+2)}{2}a^n\frac{x_ix_jx_k}{r^{(n+2)}}\left[1-\left(\frac{a}{r}\right)^2\right], \tag{3}$$

$$P_{jk} = -\mu(n+2)a^n\frac{x_jx_k}{r^{(n+2)}}\,, \tag{4}$$

where $r = |\mathbf{x}|$ and μ is the fluid viscosity. Substituting $n = 2$ or $n = 3$, one recovers the well-known results for a cylinder or sphere. The case $n = 1$ is degenerate because E_{ij} (being traceless) must be zero; there can be no straining motion in one dimension.

To compute the effective viscosity of a dilute suspension of hyperspheres we form the volume average of the stress tensor σ_{ij} (Batchelor 1970) to obtain

$$\langle\sigma_{ij}\rangle = \text{I.T.} + 2\mu\langle E_{ij}\rangle + \bar{N}\langle S_{ij}\rangle\,, \tag{5}$$

where $\langle\ \rangle$ denotes volume average, $\bar{N}$ is the number density hyperspheres, and $\langle S_{ij}\rangle = 1/N\, \Sigma_{\alpha=1}^{N} S_{ij}$ is the average extra particle stress; the sum being over all particles in the volume. I.T. stands for an isotopic term which is of no importance. For rigid particles the extra particle stress S_{ij} is given by

$$S_{ij} = \int \left\{ \sigma_{ik} x_j n_k - \frac{1}{3}\delta_{ij}\sigma_{lk} x_l n_k \right\} dS, \qquad [6]$$

the integration being over the hypersphere surface. A direct calculation of [6] gives

$$S_{ij} = \mu(n+2)\frac{a^n}{n}\frac{2\pi^{n/2}}{\Gamma\left(\frac{1}{2}n\right)} E_{ij}$$

where $\Gamma(x)$ is the gamma function. Whence, the effective viscosity

$$\mu_{\text{eff}} = \mu\left(1 + \frac{n+2}{2}\phi\right) \qquad [7]$$

where

$$\phi = \bar{N}\frac{a^n}{n}\frac{2\pi^{n/2}}{\Gamma\left(\frac{1}{2}n\right)}$$

is the volume fraction of hyperspheres. $n = 3$ gives the Einstein correction of $5/2\phi$ (Einstein 1906), and $n = 2$ gives the result for rigid cylinders first reported by Belzons *et al.* (1981).

REFERENCES

BATCHELOR, G. K. 1970 The stress system in a suspension of force-free particles. *J. Fluid Mech.* **41**, 545–570.

BELZONS, M., BLANC, R., BOUILLOT, J-L. & CAMOIN, C. 1981 Viscosité d'une suspension diluée et bidimensionnelle de sphères. *C.R. Acad. Sc. Paris* **292 II**, 939–944.

BRENNER, H. 1981 The translational and rotational motions of an n-dimensional hypersphere through a viscous fluid at small Reynolds numbers. *J. Fluid Mech.* **111**, 197–215.

EINSTEIN, A. 1906 Eine neue Bestimmung der Moleküldimensionen. *Annln. Phys.* **19**, 298–306 (and **34**, 591–592).